

Detached-Eddy Simulation Based on the v^2 - f Model

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Abstract: Detached-eddy simulation (DES) based on the v^2 - f Reynolds-averaged Navier-Stokes (RANS) model is developed and tested. The v^2 - f model incorporates the anisotropy of near-wall turbulence, which is absent in other RANS models commonly used in the DES community. The v^2 - f RANS model is modified in order that the proposed v^2 - f -based DES formulation reduces to a transport equation for the subgrid-scale kinetic energy in isotropic turbulence. First, three coefficients in the elliptic relaxation equation are modified, which is tested in channel flows with friction Reynolds number up to 2000. Then, the proposed v^2 - f DES model formulation is derived. The constant, C_{DES} , required in the DES formulation was calibrated by simulating both decaying and statistically-steady isotropic turbulence. After C_{DES} was calibrated, the v^2 - f DES formulation is tested for flow around a circular cylinder at a Reynolds number of 3900, in which case turbulence develops after separation. Simulations indicate that this model represents the turbulent wake nearly as accurately as the dynamic Smagorinsky model. Spalart-Allmaras-based DES is also included in the cylinder flow simulation for comparison.

Keywords: detached-eddy simulation, v^2 - f model, hybrid RANS/LES.

1 Introduction

Detached-eddy simulation (DES) [1, 2, 3] is a hybrid RANS/LES approach that performs RANS in attached regions and LES in detached regions using a single model. In the LES region, the length scale of the model is set proportional to the grid size Δ . The RANS model thereby becomes an LES model.

Although the Spalart-Allmaras (SA) model has been widely used for DES [3], its near-wall damping does not distinguish between velocity components. In contrast, the v^2 - f formulation [4] models the suppression of wall normal velocity fluctuation caused by non-local pressure-strain effects. This anisotropy has been shown to improve prediction of separation and reattachment [5, 6]. Among the few works where both v^2 - f and SA RANS were compared, Iaccarino et al. [6] predicted the pressure coefficient C_p with v^2 - f more accurately than with SA for flow over a 2D hump with steady-suction flow control. Constantinescu et al. [7] achieved an accurate friction coefficient C_f on a sphere after laminar separation with v^2 - f but an inaccurate prediction with the k - ω , k - ϵ , and SA RANS models.

The v^2 - f -based DES model developed here incorporates more flow physics than SA-based DES. The SA model uses the minimum distance to the wall as the turbulence length scale which is not necessarily accurate at or near separation. The v^2 - f model, on the other hand, computes a length scale based on flow properties (namely, the kinetic energy k , the dissipation rate ϵ , and the kinematic viscosity ν). In addition, the present v^2 - f DES model gives a transport equation for subgrid-scale (sgs) kinetic energy which is less empirical than the sgs viscosity transport equation used in SA-based DES.

A brief summary of the proposed v^2 - f DES methodology is as follows. The v^2 - f RANS model [4] has transport equations for k , ϵ , and $\overline{v^2}$ together with an elliptic relaxation equation for a function f . These equations contain length and time scales L and T . The choice between RANS and LES modes is made by setting these scales appropriately. When the grid is fine enough to capture large-scale turbulent eddies, in particular, when $C_{DES}\Delta < k^{3/2}/\epsilon$, LES mode is selected. Otherwise, RANS mode is selected. As suggested by Spalart et al. [1], the coefficient C_{DES} is chosen to match the correct energy spectrum in isotropic turbulence. In LES mode, the time scale is set to $T = C_{DES}\Delta/\sqrt{k}$ where k now represents the sgs kinetic

energy. Although the k equation is sufficient for purely LES simulations, the other three equations are required for the RANS mode. The v^2 - f model [4] is modified so that the entire set of equations reduces in LES mode to a transport equation for the sgs k . In particular, coefficients in the elliptic relaxation equation are modified so that v^2 becomes statistically $(2/3)k$ in the limit of LES for isotropic turbulence.

The proposed v^2 - f DES model is described in section 2. In section 3, results for this model are compared with the dynamic Smagorinsky model [8] and SA-based delayed DES [2] using the same flow solver and grid for flow past a cylinder at Reynolds number based on the diameter $Re_D = 3900$. This cylinder flow is a good test case with grids appropriate for non-dissipative schemes. This test is not chosen to determine which DES model is better than others, since it is expected that all DES models will behave like the Smagorinsky model for this flow. Conclusions are presented in section 4.

2 v^2 - f DES Model

The DES model proposed is based on Lien et al.'s version of the v^2 - f RANS model [4]. The quantity v^2 should be interpreted as a velocity scalar, not a Reynolds stress tensor component, and the function f makes v^2 behave like the wall normal component of the Reynolds stress $\langle u'_n u'_n \rangle$ [5]. Another meaning that we believe should be imposed on this scalar is that in isotropic turbulence v^2 should be the average of the normal components of the Reynolds stress. This assumption allows us to formulate a DES model that reduces to a transport equation for sgs- k in isotropic turbulence. This derivation starts from the v^2 - f RANS Eqs.(1)-(5). To reach the final formulation (Eqs.(18)-(22)), the formulations of sgs- k LES (Eqs.(12)-(14)) and k - ϵ DES (Eqs.(15)-(17)) are derived on the way.

The v^2 - f RANS model of Lien et al. [4] is

$$\nu_t = c_\mu \overline{v^2} T_{\text{RANS}} \quad (1)$$

$$\partial_t k + U_j \partial_j k = \mathcal{P} - \epsilon + \partial_j [(\nu + \nu_t) \partial_j k] \quad (2)$$

$$\partial_t \epsilon + U_j \partial_j \epsilon = \frac{c_{\epsilon 1} \mathcal{P} - c_{\epsilon 2} \epsilon}{T_{\text{RANS}}} + \partial_j \left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon} \right) \partial_j \epsilon \right] \quad (3)$$

$$\partial_t \overline{v^2} + U_j \partial_j \overline{v^2} = k f - 6 \frac{\overline{v^2}}{k} \epsilon + \partial_j [(\nu + \nu_t) \partial_j \overline{v^2}] \quad (4)$$

$$c_L^2 L_{\text{RANS}}^2 \nabla^2 f - f = \frac{1}{T_{\text{RANS}}} \left[(c_1 - 6) \frac{\overline{v^2}}{k} - \frac{2}{3} (c_1 - 1) \right] - c_2 \frac{\mathcal{P}}{k}, \quad (5)$$

where the time and length scales are defined by

$$T_{\text{RANS}} = \min \left[\max \left[\frac{k}{\epsilon}, c_T \left(\frac{\nu}{\epsilon} \right)^{1/2} \right], \frac{0.6k}{\sqrt{6} C_\mu \overline{v^2} |S_{ij}|} \right], \quad L_{\text{RANS}} = \max \left[\min \left[\frac{k^{3/2}}{\epsilon}, \frac{k^{3/2}}{\sqrt{6} C_\mu \overline{v^2} |S_{ij}|} \right], c_\eta \left(\frac{\nu^3}{\epsilon} \right)^{1/4} \right]. \quad (6)$$

Note that the realizability constraints of Durbin [9] are imposed to both scales [10]. The turbulence production $\mathcal{P} = 2\nu_t |S_{ij}|^2$, and the strain rate tensor $S_{ij} = \frac{1}{2} (\partial_j U_i + \partial_i U_j)$. The coefficients are given by

$$\begin{aligned} c_\mu &= 0.22, \quad c_{\epsilon 1} = 1.4 \left(1 + 0.045 \sqrt{\frac{k}{v^2}} \right), \quad c_{\epsilon 2} = 1.9, \quad \sigma_\epsilon = 1.3, \\ c_1 &= 1.4, \quad c_2 = 0.3, \quad c_T = 6, \quad c_L = 0.23, \quad c_\eta = 70. \end{aligned} \quad (7)$$

Wall boundary conditions are

$$k_w = 0, \quad \epsilon_w = \nu \partial_n^2 k|_w, \quad \overline{v^2}_w = 0, \quad f_w = 0, \quad (8)$$

where ∂_n is the wall-normal gradient. Note that in Eq.(5), we have chosen to keep c_L outside of the definition of L_{RANS} for a later purpose, unlike Lien et al. [4].

To begin with, we modify the three coefficients in the elliptic relaxation equation (5) as follows,

$$c_2 = c_{2,0} + \frac{1}{2}(2/3 - c_{2,0}) \left[1 + \tanh \left\{ c_{2,1}(\overline{v^2}/k - c_{2,2}) \right\} \right], \quad c_{2,0} = 0.3, \quad c_{2,1} = 50, \quad c_{2,2} = 0.55 \quad (9)$$

$$c_L = 0.115, \quad c_\eta = 140 \quad (10)$$

The value of $c_{2,0}$ is the unmodified value of c_2 . The other parameters $c_{2,1}$ and $c_{2,2}$ are chosen for the modified $c_2 \rightarrow 2/3$ when $\overline{v^2}/k \gtrsim 0.6$, as shown in Fig. 1. This increases the rapid distribution term (the last term in Eq. (5)) so that it reduces to the isotropization of production (IP) model [11] when $c_2 = 2/3$. The parameters $c_{2,1}$ and $c_{2,2}$ are not optimized. Effects of these parameters on a turbulence wake behind a cylinder are discussed in section 3. The modification of c_2 allows $\overline{v^2}$ to be statistically $2/3k$ when this model is used as an sgs model for isotropic turbulence, i.e., $\langle \overline{v^2} \rangle = (2/3) \langle k \rangle$.

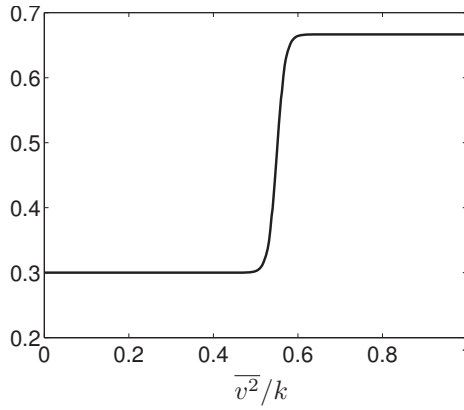


Figure 1: The modified c_2 of the present v^2 - f RANS model

To explain how the $\overline{v^2}$ equation (4) becomes $2/3$ of the k equation (2), let us approximate f as $\tilde{f}$ where $\tilde{f}$ is defined by

$$\begin{aligned} \tilde{f} &\equiv \frac{\epsilon}{k} \left[(c_1 - 6) \frac{\overline{v^2}}{k} - \frac{2}{3}(c_1 - 1) \right] + c_2 \frac{\mathcal{P}}{k} \\ &= 5 \frac{\overline{v^2}}{k^2} \epsilon + c_2 \frac{\mathcal{P}}{k}, \quad \text{if } \overline{v^2}/k = 2/3 \end{aligned} \quad (11)$$

This approximation is valid for $y^+ \gtrsim 10$ in channel flow, as shown in Fig. 2. The channel flow simulation is discussed in detail later. Substituting this approximation into Eq. (4), the right-hand side of the $\overline{v^2}$ equation (4) becomes $2/3$ of the k equation (2) because $c_2 \rightarrow 2/3$. It was found that the modified c_2 (Eq. (9)) required c_L and c_η to also be modified to yield acceptable results for channel flow. The value of the product $c_L c_\eta = 16.1$ is the same in order to keep the Laplacian term near the wall the same as in the unmodified form and to reduce it away from wall.

As briefly mentioned above, to validate the modified coefficients of the elliptic relaxation equation, fully developed turbulent channel flows with friction Reynolds numbers Re_τ up to 2000 are simulated in RANS mode. Figure 3 shows that this modification maintains the performance of the unmodified RANS model with a slightly better prediction of $\overline{v^2}$ near the center.

In the DES approach, a RANS model switches to an LES model based on a comparison of length scales. The LES length scale is determined by the grid size Δ , so only a time scale equation is needed for an sgs model. We would like the v^2 - f DES model to reduce to an sgs- k equation model similar to Yoshizawa's model [12] given that for isotropic turbulence $\langle \overline{v^2} \rangle$ becomes the average of the sgs normal stress i.e., $\langle \overline{v^2} \rangle = (2/3) \langle k \rangle$.

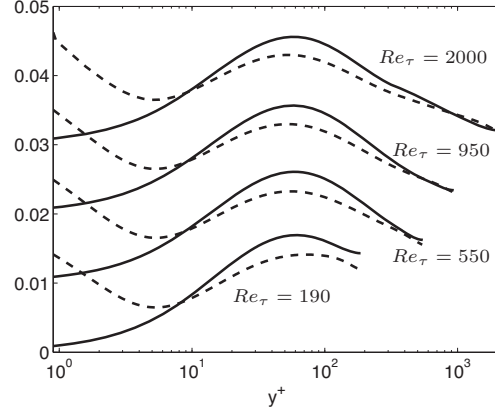


Figure 2: The function f^+ (—) and its approximation $\tilde{f}^+ = 5\overline{v^2}^+ \epsilon^+ / k^{+2} + c_2 \mathcal{P}^+ / k^+$ i.e., Eq.(11) (- -) in the present v^2 - f RANS for the channel. The profiles of $Re_\tau = 550$, $Re_\tau = 950$ and $Re_\tau = 2000$ are shifted upward along the vertical axis by 0.01, 0.02 and 0.03, respectively. The plus superscript indicates wall units.

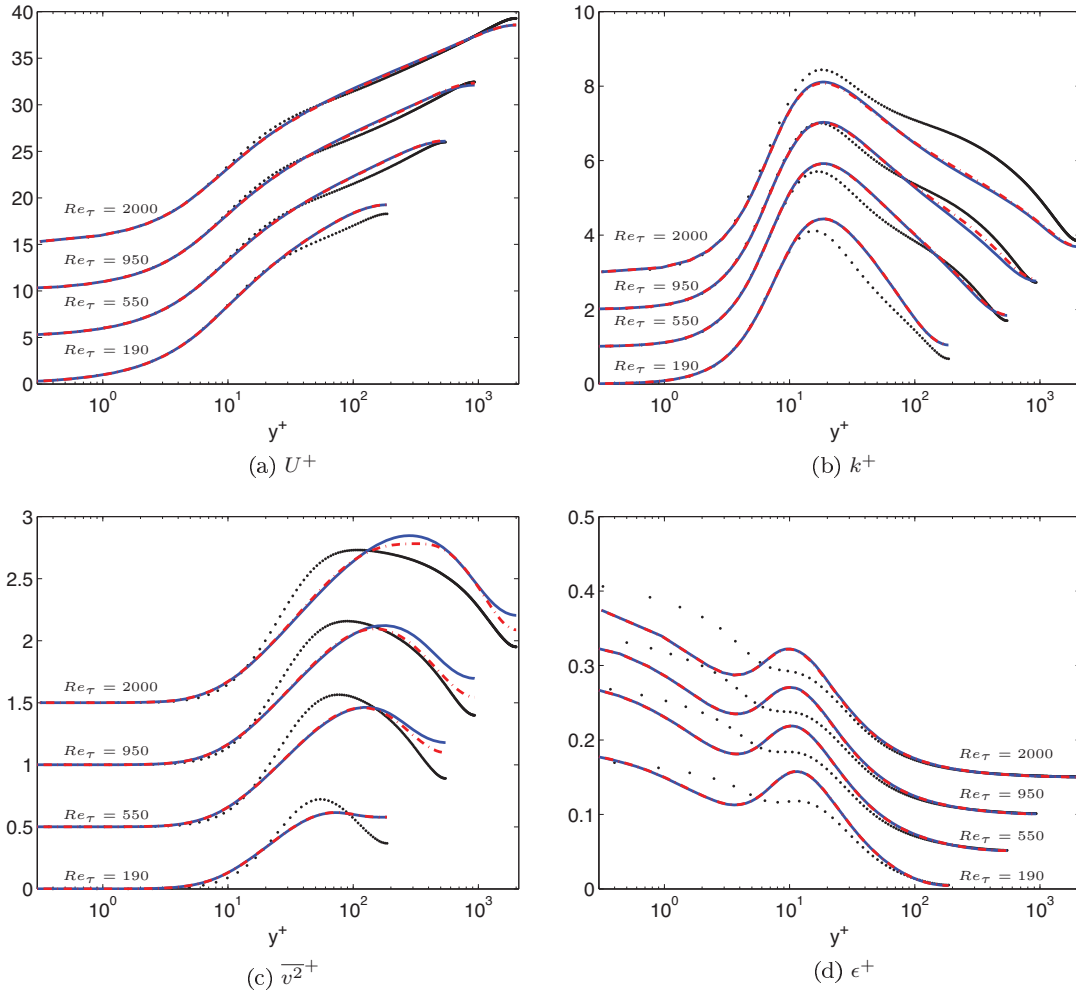


Figure 3: Comparison of DNS's (dots) and RANS with the unmodified [4] (—) and the present v^2 - f model (- · -) in turbulent channel flows. The plus superscript indicates wall units.

An sgs- k LES model is

$$\nu_t = c_\mu \frac{2}{3} k T_{\text{LES}}, \quad T_{\text{LES}} = \frac{k}{\epsilon_{\text{LES}}} \quad (12)$$

$$\partial_t k + U_j \partial_j k = \mathcal{P} - \epsilon_{\text{LES}} + \partial_j [(\nu + \nu_t) \partial_j k] \quad (13)$$

$$\epsilon_{\text{LES}} = \frac{k^{3/2}}{L_{\text{LES}}}, \quad L_{\text{LES}} = C_{\text{DES}} \Delta \quad (14)$$

When the flow is in equilibrium i.e., $\mathcal{P} = \epsilon_{\text{LES}}$, a Smagorinsky-like eddy viscosity $\nu_t \propto |S_{ij}| \Delta^2$ is obtained.

As an intermediate step towards the v^2 - f DES model, a simplified k - ϵ based DES model (without damping functions) is presented. Incorporating the dissipation equation into the DES context yields

$$\nu_t = c_\mu \frac{2}{3} k T_{\text{DES}}, \quad T_{\text{DES}} = L_{\text{DES}} / \sqrt{k}, \quad L_{\text{DES}} = \min[k^{3/2}/\epsilon, \underbrace{C_{\text{DES}} \Delta}_{L_{\text{LES}}} \quad (15)$$

$$\partial_t k + U_j \partial_j k = \mathcal{P} - \epsilon_{\text{DES}} + \partial_j [(\nu + \nu_t) \partial_j k], \quad \epsilon_{\text{DES}} = k^{3/2} / L_{\text{DES}} \quad (16)$$

$$\partial_t \epsilon + U_j \partial_j \epsilon = \frac{c_{\epsilon 1} \mathcal{P} - c_{\epsilon 2} \epsilon}{T_{\text{DES}}} + \partial_j \left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon} \right) \partial_j \epsilon \right] \quad (17)$$

If the RANS length scale is selected, then $T_{\text{DES}} = k/\epsilon$ and $\epsilon_{\text{DES}} = \epsilon$ as expected. Otherwise, $T_{\text{DES}} = k/\epsilon_{\text{LES}}$ which is the same as T_{LES} in Eq.(12), and Eqs.(15)-(17) reduce to Eqs.(12)-(14).

Finally, to present the full v^2 - f DES model, the v^2 and f equations are included in the complete set of equations:

$$\nu_t = c_\mu \overline{v^2} T_{\text{DES}} \quad (18)$$

$$\partial_t k + U_j \partial_j k = \mathcal{P} - \epsilon_{\text{DES}} + \partial_j [(\nu + \nu_t) \partial_j k] \quad (19)$$

$$\partial_t \epsilon + U_j \partial_j \epsilon = \frac{c_{\epsilon 1} \mathcal{P} - c_{\epsilon 2} \epsilon}{T_{\text{DES}}} + \partial_j \left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon} \right) \partial_j \epsilon \right] \quad (20)$$

$$\partial_t \overline{v^2} + U_j \partial_j \overline{v^2} = k f - 6 \frac{\overline{v^2}}{k} \epsilon_{\text{DES}} + \partial_j [(\nu + \nu_t) \partial_j \overline{v^2}] \quad (21)$$

$$c_L^2 L_{\text{DES}}^2 \nabla^2 f - f = \frac{1}{T_{\text{DES}}} \left[(c_1 - 6) \frac{\overline{v^2}}{k} - \frac{2}{3} (c_1 - 1) \right] - c_2 \frac{\mathcal{P}}{k} \quad (22)$$

$$\text{RANS mode : if } k^{3/2}/\epsilon < L_{\text{LES}}, \text{ then } L_{\text{DES}} = L_{\text{RANS}}, \quad T_{\text{DES}} = T_{\text{RANS}}, \text{ and } \epsilon_{\text{DES}} = \epsilon \quad (23)$$

$$\text{LES mode : otherwise, } L_{\text{DES}} = C_{\text{DES}} \Delta, \quad T_{\text{DES}} = C_{\text{DES}} \Delta / \sqrt{k}, \text{ and } \epsilon_{\text{DES}} = k^{3/2} / (C_{\text{DES}} \Delta), \quad (24)$$

where T_{RANS} and L_{RANS} are defined in Eq.(6). Note that $(2/3)k$ appears in Eqs.(12) and (15) but $\overline{v^2}$ appears in Eq.(18). For an LES simulation of isotropic turbulence, $\overline{v^2}$, if interpreted as the sgs average normal stress, should be statistically $(2/3)k$. The modified c_2 in Eq.(9) provides this behavior. Simulations indicate that the unmodified $c_2 = 0.3$ yields $\langle \overline{v^2} \rangle / \langle k \rangle \simeq 0.4$ for isotropic turbulence.

Note that the Kolmogorov length scale present in Eq. (6) is not considered in the length comparison in Eqs. (23) and (24) because it is expected that, near the RANS and LES transition (or away from the wall), the RANS length scale will be $k^{3/2}/\epsilon$. However, discontinuity in length or time scale may occur if the Kolmogorov length scale happens to be the RANS scale at the RANS/LES transition. This would occur in low- Re flows. Another way to avoid such a discontinuity is maybe to use L_{RANS} for the length scale comparison. This alternative is not considered here because it turns off the realizability constraints in Eq. (6) causing the stagnation point anomaly [9] near the front stagnation point on a cylinder. Flow simulation over a cylinder is discussed in section 3.

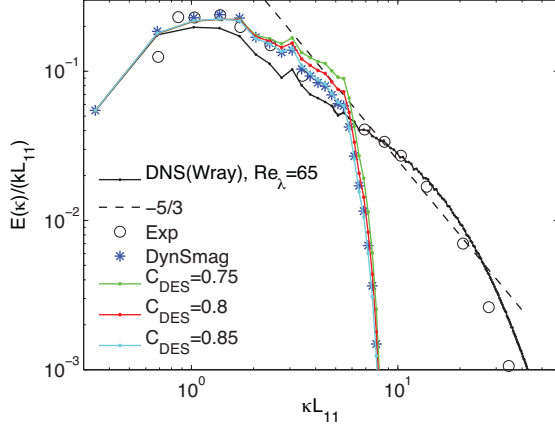
The proposed DES model is implemented in the Stanford incompressible Navier-Stokes solver CDP (v2.3). This code is based on finite-volume spatial discretization with second-order accuracy and a second-order implicit fractional step method [13, 14]. CDP employs a novel collocated formulation to conserve mass,

momentum and approximately kinetic energy (in the inviscid limit) on a hybrid unstructured mesh. The second-order central difference scheme is used for both the convection and diffusion terms of the momentum equation. The scalar transport equations (19)-(21) are time advanced implicitly. To achieve numerical stabilization for the scalar equations, which are significantly controlled by the source terms, the convective terms in these equations are discretized with the 1st-order upwind scheme. The diffusion term in each scalar equation is discretized with the central scheme. In the test of isotropic turbulence, when this upwind scheme is used for the scalar convection term only, it yields the identical energy spectrum compared to use of the central scheme for all the terms. An upwind-biased scheme is never used for the momentum equation in this study.

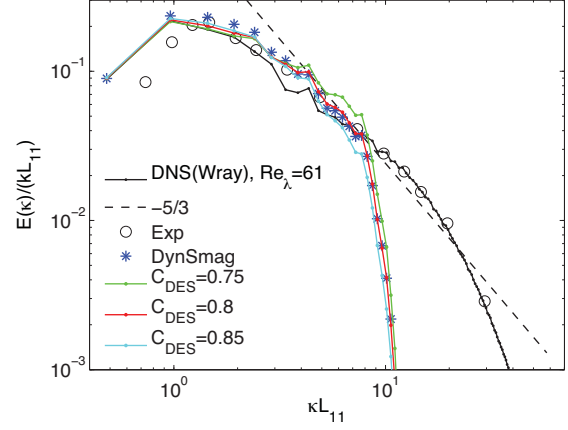
To calibrate C_{DES} , decaying isotropic turbulence is considered. The initial field uses 512^3 DNS data of Wray [15] at $Re_\lambda = 105$ sampled down to 32^3 . The computational cube has sides $L_{\text{box}} = 2\pi$ with periodic boundary conditions in the three directions. Quantities in the DES model are initialized by running for some time with a frozen initial velocity field. The flow is allowed to run about 3.7 and 10.7 large-eddy turnover times at which point $Re_\lambda = 65$ and $Re_\lambda = 61$, respectively. The large-eddy turnover timescale is determined by the longitudinal length scale L_{11} and the r.m.s. velocity fluctuation u_{rms} computed from the DNS data at $Re_\lambda = 105$ (the initial state).

Figs. 4 (a) and (b) show the energy spectrum with three values of C_{DES} . The v^2 - f DES simulation with $C_{\text{DES}} = 0.8$ agrees very well with DNS [15] and experiments [16] up to the cut-off wave number at the later instant when $Re_\lambda = 61$ (Fig. 4 (b)). v^2 - f DES gives almost exactly the same energy spectrum as produced by the dynamic Smagorinsky model (Figs. 4 (c) and (d)). In Figs. 4 (e) and (f), all three formulations, sgs- k LES, k - ϵ DES, and v^2 - f DES, give the same energy spectrum with the same C_{DES} . This indicates that the v^2 - f DES formulation indeed reduces to the others. The coefficient $C_{\text{DES}} = 0.8$ is used for the rest of the paper.

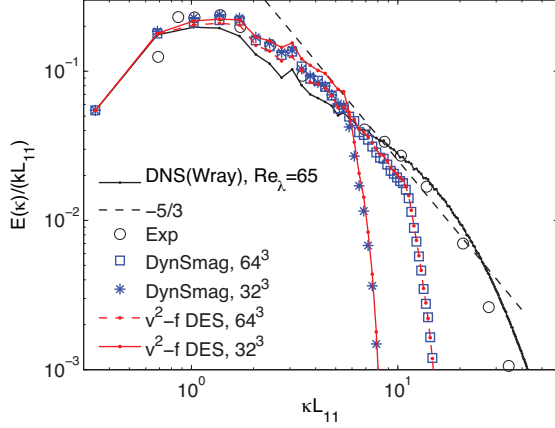
Because it allows a higher Reynolds number and a wider inertial range, forced isotropic turbulence at $Re_\lambda = 98$ is also considered using the stochastic forcing of Eswaran and Pope [17]. The computational cube has the same size $L_{\text{box}} = 2\pi$ as the decaying case with the same periodic conditions. The radius of the sphere of forced wavenumbers is $K_F = \sqrt{8}$, giving a total 92 forced modes. The forcing amplitude and the forcing autocorrelation time scale are chosen as $\sigma = 0.3572$ and $T_L = 0.4312$, respectively, following the nomenclature of Ref. [17]. Initial conditions are not relevant, since the flow is driven to a statistical steady state. The grid is 128^3 for the present DNS, and two grids of 16^3 and 32^3 are used for both LES with the dynamic Smagorinsky model and v^2 - f DES. Statistics were obtained by averaging over samples collected over 30 eddy turnover times in the steady state which is reached after 10 eddy turnover times from the start. The velocity-derivative skewness $S = 1/3 \left[\langle (\partial_i u_i)^3 \rangle / \langle (\partial_j u_j)^2 \rangle^{3/2} \right]$ has values (~ 0.45) typical of isotropic turbulent at $Re_\lambda \sim 100$ [18]. The v^2 - f DES simulation gives almost exactly the same energy spectrum as produced by the dynamic Smagorinsky model (Fig. 5). Forced isotropic turbulence had not been used for the C_{DES} calibration in the DES community. This simulation shows that the forced case is an alternative for this calibration.



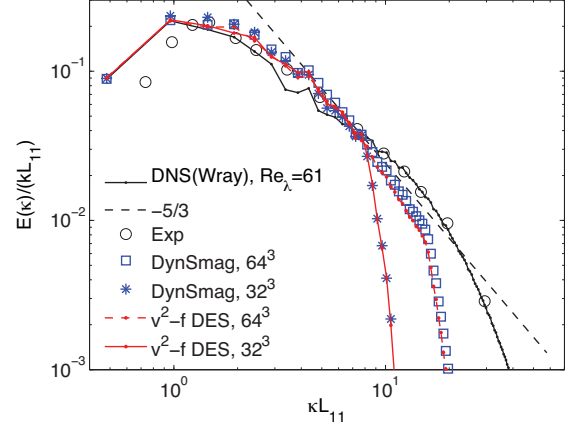
(a) v^2 -f DES with three different C_{DES} at $Re_\lambda = 65$



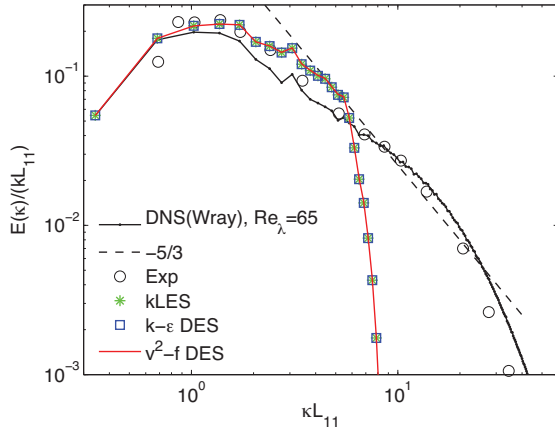
(b) v^2 -f DES with three different C_{DES} at $Re_\lambda = 61$



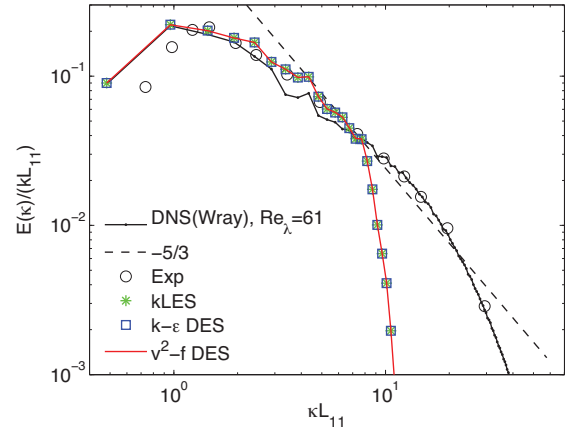
(c) LES and v^2 -f DES with $C_{DES} = 0.8$ at $Re_\lambda = 65$



(d) LES and v^2 -f DES with $C_{DES} = 0.8$ at $Re_\lambda = 61$



(e) sgs- k LES, k - ϵ DES and v^2 -f DES with $C_{DES} = 0.8$



(f) sgs- k LES, k - ϵ DES and v^2 -f DES with $C_{DES} = 0.8$

Figure 4: Energy spectrum of decaying isotropic turbulence. The proposed v^2 -f DES model is simulated with three different values of C_{DES} on 32^3 in (a) and (b). The curve labeled “-5/3” shows $E(\kappa) = C\epsilon^{2/3}\kappa^{-5/3}$ with the Kolmogorov constant $C = 1.5$. “DynSmag” indicates LES with the dynamic Smagorinsky model. The coefficient $C_{DES} = 0.8$ is used for sgs- k LES, k - ϵ DES and v^2 -f DES on 32^3 in (e) and (f).

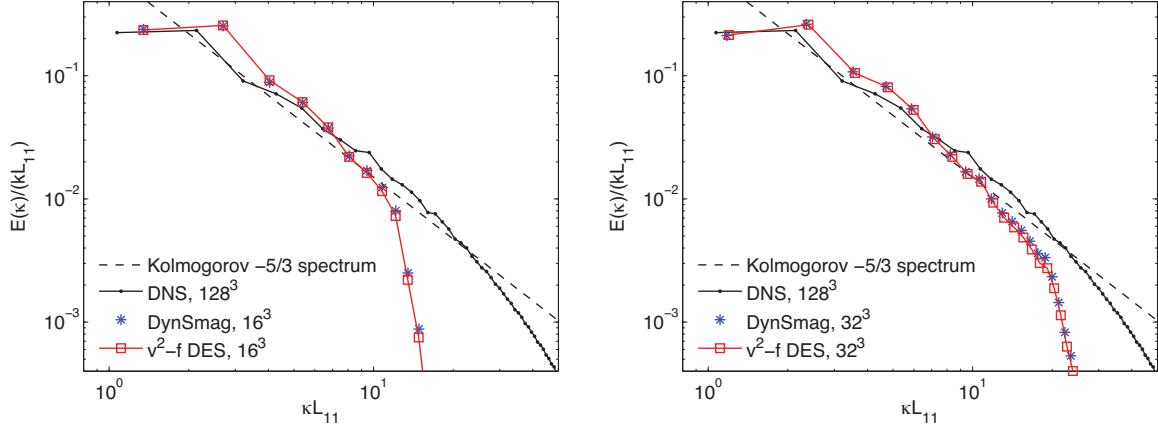


Figure 5: Energy spectrum of forced isotropic turbulence of $Re_\lambda = 98$. Shown are the present simulations of DNS, LES with the dynamic Smagorinsky model, and v^2 -f DES with $C_{DES} = 0.8$ on the 16^3 grid (left) and the 32^3 grid (right).

3 Results: Flow over a circular cylinder

To test the LES mode of the proposed v^2 -f DES model, flow over a circular cylinder is chosen. The Reynolds number based on diameter D is $Re_D = 3900$. The DES model coefficient C_{DES} , the only coefficient which has a degree of freedom, is determined in isotropic turbulence. Therefore, it is imperative to test the DES model in anisotropic turbulence. This flow over a circular cylinder is a good test case, because (1) turbulence in the wake is anisotropic, and (2) this flow has been extensively studied both experimentally [19, 20] and numerically [14, 21, 19, 20]. Kravchenko and Moin [21] used a Galerkin B-spline method with the dynamic Smagorinsky model to simulate this flow with good agreement of mean flow quantities with experimental data available at the time. Mahesh et al. [14] simulated this flow with the dynamic Smagorinsky model to validate the numerical method used in the CDP code. Dong et al. [19] performed DNS using a spectral element scheme on an unstructured grid and compared it with their PIV experiments. Recently, Parnaudeau et al. [20] performed an extensive study in order to address the lack of consensus in the literature for turbulence statistics immediately behind the cylinder. Parnaudeau et al. [20] simulated this flow with the dynamic Smagorinsky model using an immersed boundary method on uniform grids, and compared the computations with their PIV and hot-wire experiments.

The cylinder flow at $Re_D = 3900$ allows us to test DES as an LES model with grids appropriate for non-dissipative schemes. Strelets [22] and Travin et al. [23] performed SA-based DES for a cylinder at $Re_D = 50000$. We attempted this higher- Re case first but discovered that the resolution used in the previous studies with dissipative upwind-biased schemes was inadequate for the present second-order non-dissipative code. Strelets used a hybrid central/upwind approximation [22], and Travin et al. used a fifth-order upwind scheme [23]. The resolution in Strelets [22] and Travin et al. [23] is even coarser than that in previous LES computations at $Re_D = 3900$ [24, 21, 14, 20]; Refs. [22, 23] used about 0.7 million points, whereas 1.3 - 4.4 million points are used in Refs. [24, 21, 14, 20]. Mittal and Moin [24] concluded that even a 5th-order upwind scheme can degrade LES computations at $Re_D = 3900$. Simulation of the $Re_D = 50000$ flow is still challenging with non-dissipative schemes. Because both $Re_D = 3900$ and 50000 are subcritical (turbulence develops after separation), the $Re_D = 3900$ flow is a good test case with grids appropriate for non-dissipative schemes. This test is not chosen to determine which DES model is better than others, since it is expected that all DES models will behave like the Smagorinsky model for this flow.

The present computation was performed on a domain whose inflow and outflow surfaces were $30D$ upstream and $35D$ downstream from the center of the cylinder, following the grid of Mahesh et al. [14]. The domain height was $50D$ and spanwise extent was πD . The size of the first grid adjacent to the cylinder is $0.0025D$ radially and $0.01D$ (or 0.57°) in the azimuthal direction θ . The quadrilateral elements are approximately $0.04D \times 0.04D$ at a distance of $2D$ from the cylinder center in the wake. The current grid

has slightly higher resolution on the wall than grids in previous LES studies [21, 14, 20] and less resolution than the DNS study [19], particularly in the azimuthal θ and spanwise z directions. There are twelve grid points inside the boundary layer at separation. Fig. 6 shows the domain and the grid. The flow field is initially uniform. Without any imposed perturbation, simulation generates 3D fluctuations due to biased round-off error. Note that this flow is naturally unstable to perturbation (see the Hopf bifurcation theory [25]). A statistically steady state is obtained after about 200 time units. Statistics were accumulated over approximately 30 vortex shedding cycles (140 time units) and over the spanwise direction. The time step is $\Delta_t = 0.001D/U_\infty$. Table 1 summarizes the computation parameters in the previous and current studies.

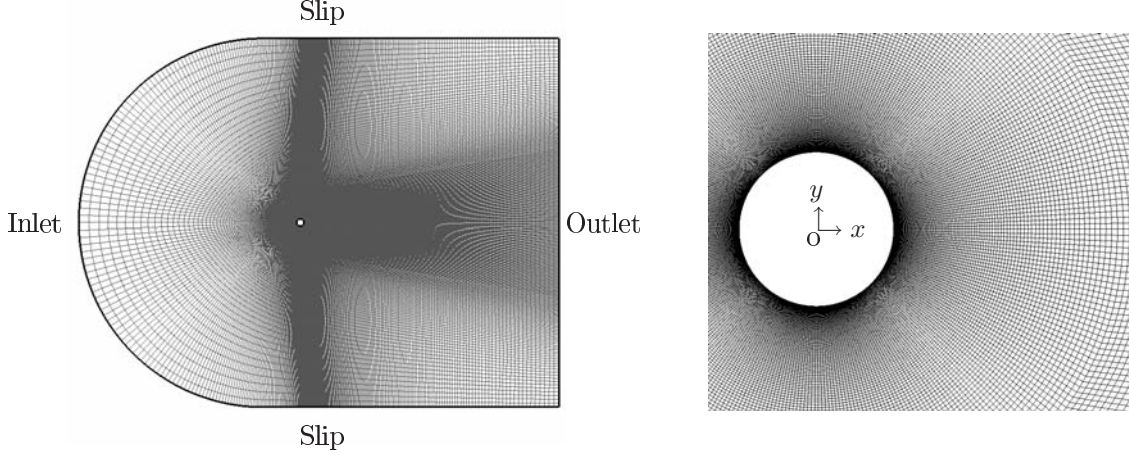


Figure 6: Computation domain (left) and grid around the cylinder (right).

Study	Kravchenko et al. [21]	Mahesh et al. [14]	Parnaudeau et al. [20]	Dong et al. [19]	Present
Domain $[D]$	$(R_d, L_z) = (30, \pi)$	$L_{x,y,z} = (75, 50, \pi)$	$L_{x,y,z} = (20, 20, \pi)$	$L_{x,y,z} = (40, 18, \pi)$	$L_{x,y,z} = (75, 50, \pi)$
Grid	$n_{r,\theta,z} = (205, 185, 48)$	$n_z = 32^*$	$n_{x,y,z} = (961, 960, 48)^\dagger$	$n_z = 128^\ddagger$	$n_z = 64^*$
$\Delta_{r,\theta} _{\text{wall}} [10^{-2}D]$	(0.25, 1.7)	(0.25, 1.2) §	(2.1, 2.1)	(0.36, 0.77) ¶	(0.25, 1.0) $^\parallel$
$\Delta_t [10^{-3}D/U_\infty]$	2.5	1	3	2	1
Simulation type	LES	LES	LES	DNS	LES, DES

Table 1: Computation parameters used for the current study and previous studies. Note *: unstructured quadrilateral grid with resolution of $0.04D$ for a distance of $1.5D$ in the wake, † : uniform mesh, ‡ : unstructured triangular grids, § : minimum values on the cylinder, ¶ : calculated from their values in viscous wall units, $^\parallel$: uniform on the cylinder.

Three turbulence models are used for the present computations: dynamic Smagorinsky LES [8], SA-based delayed DES (SA-DDES) [2] and the proposed v^2 - f DES model. The SA equation without the trip term [26], called SA-noft2 in the turbulence community, is selected here because this version can use lower values of $\tilde{\nu}$ at freestream boundary [27]. The low-Reynolds number correction of Spalart et al. [2] is required for SA-DDES to compensate for unexpected activation of the wall functions possibly due to either low Reynolds number or very fine grids. This SA-DDES was also implemented in CDP. Using $C_{\text{DES}} = 0.65$ for SA-DDES gives the correct energy spectrum in isotropic turbulence (data not shown). $\Delta = \max(\Delta_i)$ in all cases. Initial and boundary conditions are listed in Table 2.

Instantaneous velocity and vorticity fields in the wake of a circular cylinder at $Re_D = 3900$ are shown in Figs. 7 and 8, respectively. Only the results of the v^2 - f DES simulation are shown in these instantaneous figures, and statistics from three models are compared later. Fig. 7 (a) shows unsteady recirculation region behind the cylinder. Alternating regions of positive and negative cross-stream velocity are shown in Fig. 7 (b). This is related to Karman vortices shown in Fig. 8. Fig. 7 (c) clearly shows fluctuating spanwise velocity which indicates three dimensional flow structures in the wake. Figs. 7 and 8 are conformable to the LES results of Kravchenko et al., i.e., Figs. 2-5 of Ref. [21], which show the similar large structures.

Mean flow fields are obtained from the simulation over approximately 30 vortex shedding cycles and over spanwise direction in the near wake of the cylinder. Fig. 9 shows the results of v^2 - f DES. The velocity deficit

Variables	All	v^2 - f DES				SA-DDES
	u_i	k	$\bar{v}^2$	ϵ	f	$\tilde{\nu}$
Initial Conditions	uniform u_∞	$10^{-3}u_\infty^2$	$(2/3) \times 10^{-3}u_\infty^2$	$10^{-3}u_\infty^3/D$	none	5ν
Wall Boundary	0	0	0	$\nu\partial_n^2 k$	0	0
Inlet Boundary	u_∞	0	0	zero gradient	zero gradient	0.5ν
Outlet Boundary	convective outlet				zero gradient	convective outlet
Slip Boundary	zero gradient					
Z planes	periodic					

Table 2: Initial and Boundary conditions in the present computations. The gradient ∂_n is in the wall normal direction.

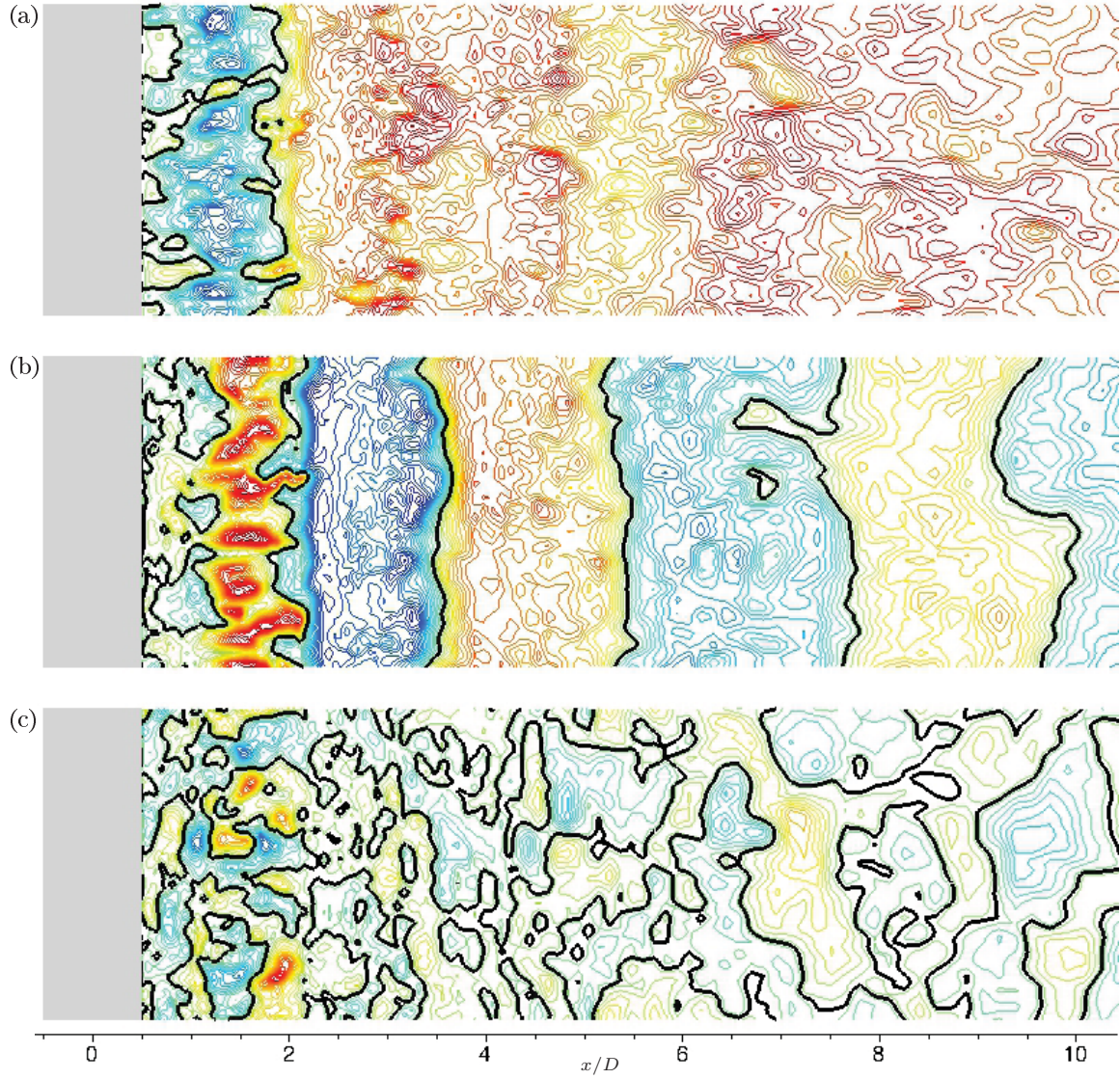


Figure 7: Instantaneous velocity fields on the spanwise mid-plane ($y = 0$) in the wake of a circular cylinder at $Re_D = 3900$ from the v^2 - f DES simulation. Shown are the normalized streamwise velocity u/u_∞ (a), the normalized cross-flow velocity v/u_∞ (b), and the normalized spanwise velocity w/u_∞ (c). The thick black lines indicate the zero value. The number of color contour levels is 52 between -1.5 (blue) and 1.5 (red).



Figure 8: Instantaneous vorticity magnitude $\omega D/u_\infty$ on a x - y plane with 16 contours from $\omega D/u_\infty = 0.5$ to $\omega D/u_\infty = 10.0$ in the simulation of v^2 - f DES.

in the wake is well shown in Fig. 9 (a) with the negative bubble near the cylinder. Separating shear layer is clearly shown in the spanwise vorticity field, Fig. 9 (b). The Reynolds shear stress $\langle u'v' \rangle$ is anti-symmetric with respect to the centerline (see Fig. 9 (c)). $\langle u'v' \rangle$ has two dominant peaks of $|\langle u'v' \rangle| = 0.119$ at $x/D = 2.08$ and two secondary peaks of $|\langle u'v' \rangle| = 0.040$ at $x/D = 1.37$. The r.m.s. streamwise velocity u_{rms} in Fig. 9 (d) shows strong gradient in the separating shear layers and two maxima associated with the vortex formation in Fig. 9 (b). These mean flow structures are qualitatively similar to Parnaudeau et al. [20]’s experimental and LES data, and Dong et al. [19]’s experimental and DNS data. Fig. 10 accumulates present simulation results with fewer contour lines for direct comparison among the three models. In general, all three models provide the similar mean flow fields with subtle differences which are emphasized by the same level of the contour lines. The simulation without any turbulence model is not included in Fig. 10 because its results are simply not correct (see Table 3).

Integral flow quantities are well predicted in the present computations with three models, as listed in Table 3. The drag coefficient $C_D = 1.00$ of v^2 - f DES matches well with previous experiment data [21] and computations [14]. This coefficient is overestimated without any turbulence model. The value $C_D = 0.965$ predicted by SA-DDES is acceptable. The base pressure coefficient $C_{p,b} = -0.928$ of v^2 - f DES is similar to that of Kravchenko et al. [21]. SA slightly underestimates $C_{p,b}$, and, without any turbulence model, $C_{p,b}$ is simply wrong. The separation point $\theta_{\text{sep}} = 87.1$ of the present dynamic-Smagorinsky LES and v^2 - f DES is in the range of the experimental data set and similar to previous LES results [21, 14, 20]. Previous studies show a broad range of the recirculation length L_R ; this value is sensitive to experimental conditions [20], and previous LES slightly underestimated this value. The current value $L_R = 1.44$ of v^2 - f DES is in the range of the experimental data. Without a turbulence model, the recirculation length is completely wrong. The Strouhal number St is the least sensitive quantity in Table. 3. The current LES, v^2 - f DES and even the simulation with no model predict the shedding frequency very accurately. SA-DDES slightly overestimates St . The minimum streamwise velocity U_{min} predicted by all the present computations is within the range of the previous data.

Fig. 11 shows mean velocity in the wake. v^2 - f DES predicts the mean streamwise velocity on the centerline observed in previous studies [21, 20] with slightly better prediction downstream $5 < x/D < 10$ than the other present computations (Fig. 11(a)). All current LES and DES’s reproduce the mean streamwise and cross-flow velocity at three locations in the wake as shown in Fig. 11(b,c). For the mean cross-flow velocity V at $x/D = 1.54$, the present LES and v^2 - f DES match closer with the experiment of Parnaudeau et al. [20] than does their LES.

Fig. 12 shows Reynolds stress profiles in the near wake. Overall, all three models predict the three components quite well, compared with LES and experimental data of Parnaudeau et al. [20]. At $x/D = 1.54$, v^2 - f DES and the present dynamic-Smagorinsky LES are slightly better than SA-DDES, particularly for $\langle u'u' \rangle$ and $\langle u'v' \rangle$. For $\langle u'u' \rangle$, v^2 - f DES and the present dynamic-Smagorinsky LES match closer with the experiment of Parnaudeau et al. [20] than does their LES, similar to V in Fig. 11 (c). Such better prediction in the present simulation is probably due to better resolution very near the cylinder; the present grid is finer

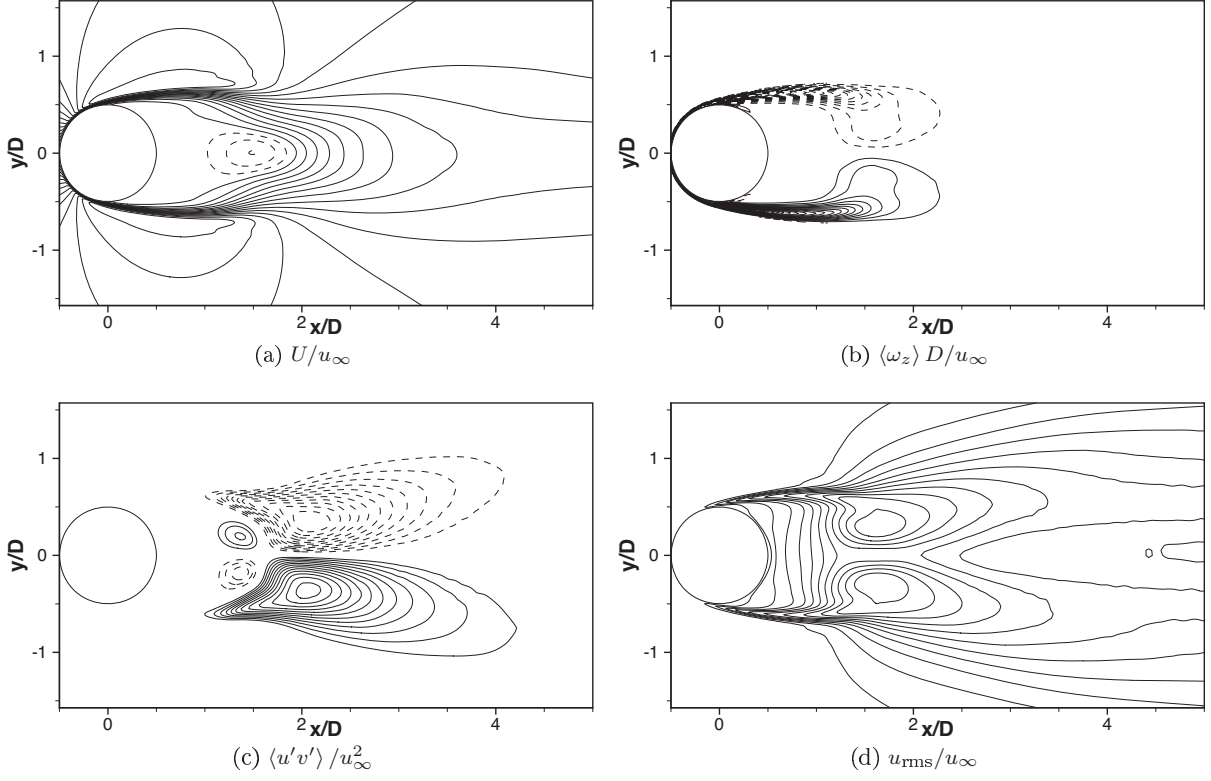


Figure 9: Mean flow fields in the simulation of v^2 - f DES: (a) the normalized streamwise velocity with contour of $U/u_\infty|_{\min} = -0.3$ and $|\Delta U/u_\infty| = 0.1$; (b) the normalized spanwise vorticity with contour of $|\langle \omega_z \rangle D/u_\infty|_{\min} = 1$ and $|\Delta \langle \omega_z \rangle D/u_\infty| = 1$; (c) the normalized Reynolds stress with contour of $|\langle u'v' \rangle / u_\infty^2|_{\min} = 0.018$ and $|\Delta \langle u'v' \rangle / u_\infty^2| = 0.009$; and (d) the normalized r.m.s. streamwise velocity with contour of $u_{rms}/u_\infty|_{\min} = 0.04$ and $|\Delta u_{rms}/u_\infty| = 0.04$. In all plots, dashed lines indicate negative contour.

	Study	C_D	$C_{p,b}$	θ_{sep}	L_R/D	St	$U_{\min}/u_\infty$
Exp.	Data set*	0.99 ± 0.05	-0.88 ± 0.05	86.0 ± 2	1.4 ± 0.1	0.215 ± 0.005	-0.24 ± 0.01
	Parnaudeau et al. [20]			88	1.51	0.208 ± 0.002	-0.34
	Dong et al. [19]				1.47		-0.252
DNS	Dong et al. [19]		-0.88		1.36	0.20	-0.291
LES	Kravchenko et al. [21]	1.04	-0.94	88.0	1.35	0.210	-0.37
	Mahesh et al. [14]	1.00		87.6	1.35	0.218	-0.31
	Parnaudeau et al. [20]			87.6	1.35	0.208	-0.31
	Present	0.994	-0.881	87.1	1.48	0.214	-0.320
DES	Present, v^2 - f DES	1.00	-0.928	87.1	1.44	0.214	-0.304
	Present, SA-DDES	0.965	-0.969	88.3	1.37	0.221	-0.283
	Present, w/o turb. model	1.18	-1.16	89.4	0.841	0.214	-0.264

Table 3: Mean flow quantities of the flow around the circular cylinder at $Re_D = 3900$: the drag coefficient C_D , the base pressure coefficient $C_{p,b}$, the separation angle from the front θ_{sep} , the recirculation length L_R , the Strouhal number St , and the minimum streamwise velocity $U_{\min}$ on the centerline. Note *: experimental data in Table II of Kravchenko et al. [21]

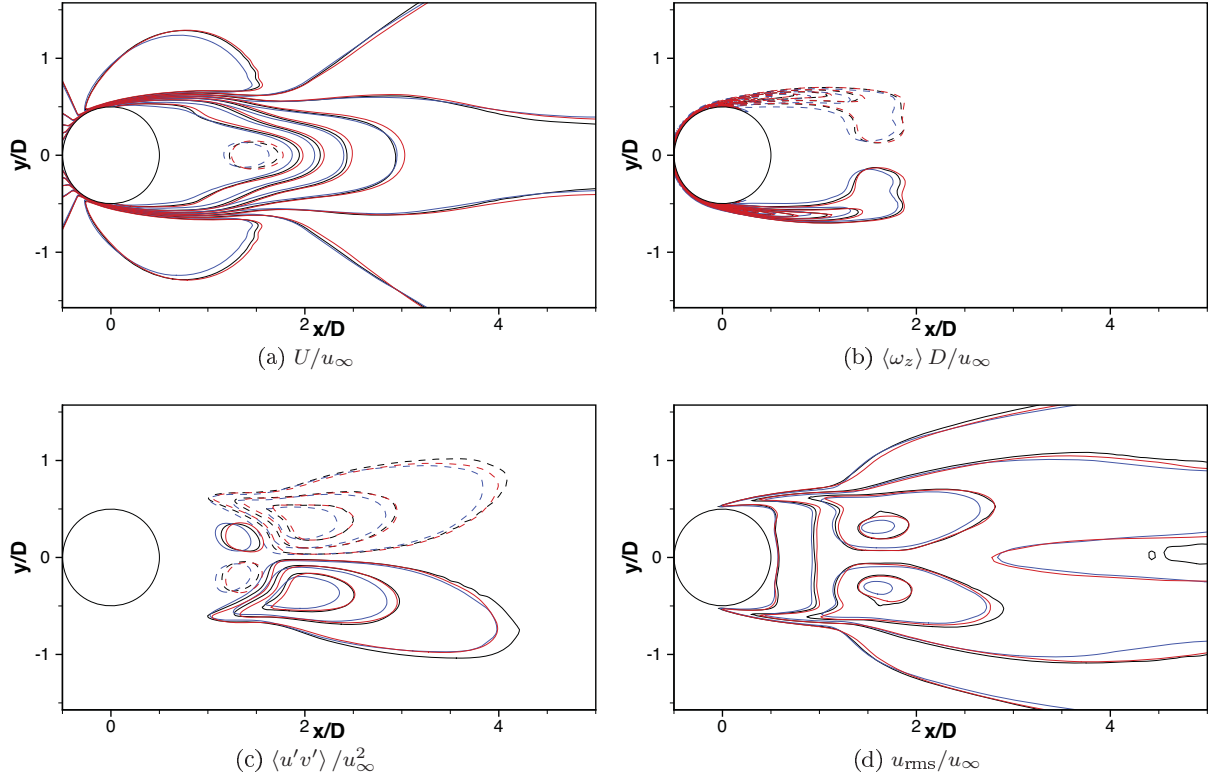


Figure 10: Mean flow fields in the simulation of v^2 - f DES (—), dynamic-Smagorinsky LES (—), and SA-DDES (—): (a) the normalized streamwise velocity with contour of $U/u_\infty|_{\min} = -0.2$ and $|\Delta U/u_\infty| = 0.2$; (b) the normalized spanwise vorticity with contour of $|\langle \omega_z \rangle D/u_\infty|_{\min} = 2$ and $|\Delta \langle \omega_z \rangle D/u_\infty| = 4$; (c) the normalized Reynolds stress with contour of $|\langle u'v' \rangle / u_\infty^2|_{\min} = 0.018$ and $|\Delta \langle u'v' \rangle / u_\infty^2| = 0.036$; and (d) the normalized r.m.s. streamwise velocity with contour of $u_{\text{rms}}/u_\infty|_{\min} = 0.08$ and $|\Delta u_{\text{rms}}/u_\infty| = 0.12$. In all plots, dashed lines indicate negative contour.

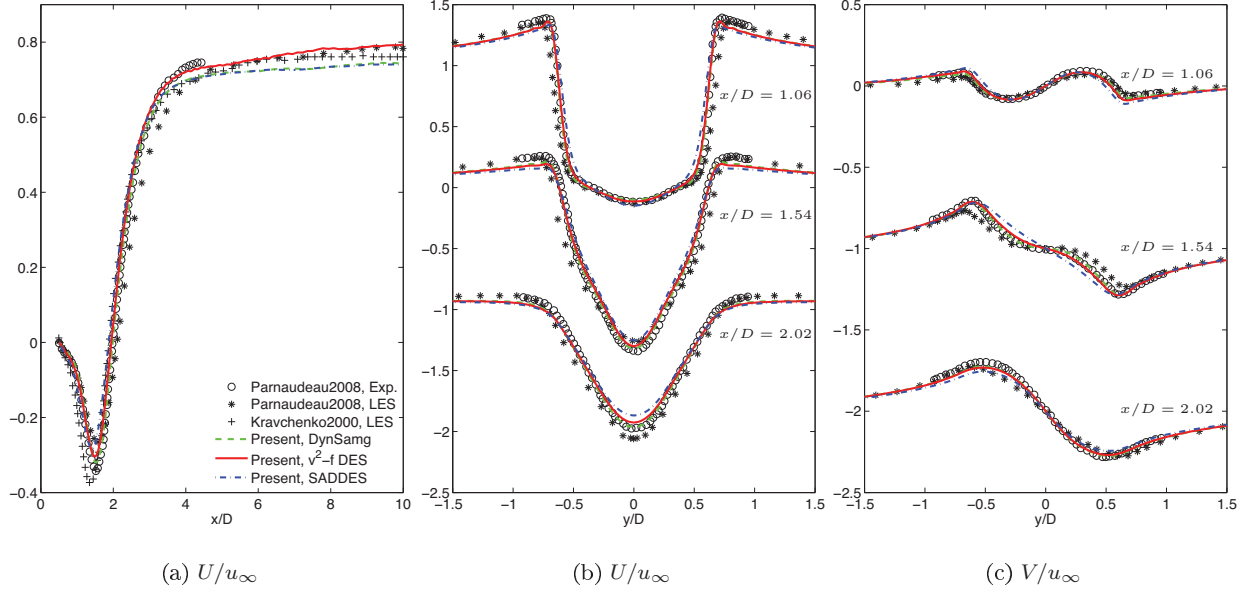


Figure 11: Mean velocity profiles in the wake of the circular cylinder at $Re_D = 3900$. Shown are (a) normalized streamwise velocity U/u_∞ on the centerline, and (b) U/u_∞ and (c) normalized cross-flow velocity V/u_∞ at three locations in the wake of a circular cylinder at $Re_D = 3900$. In (b) and (c), profiles at $x/D = 1.54$ and $x/D = 2.02$ are vertically shifted down by 1 and 2, respectively.

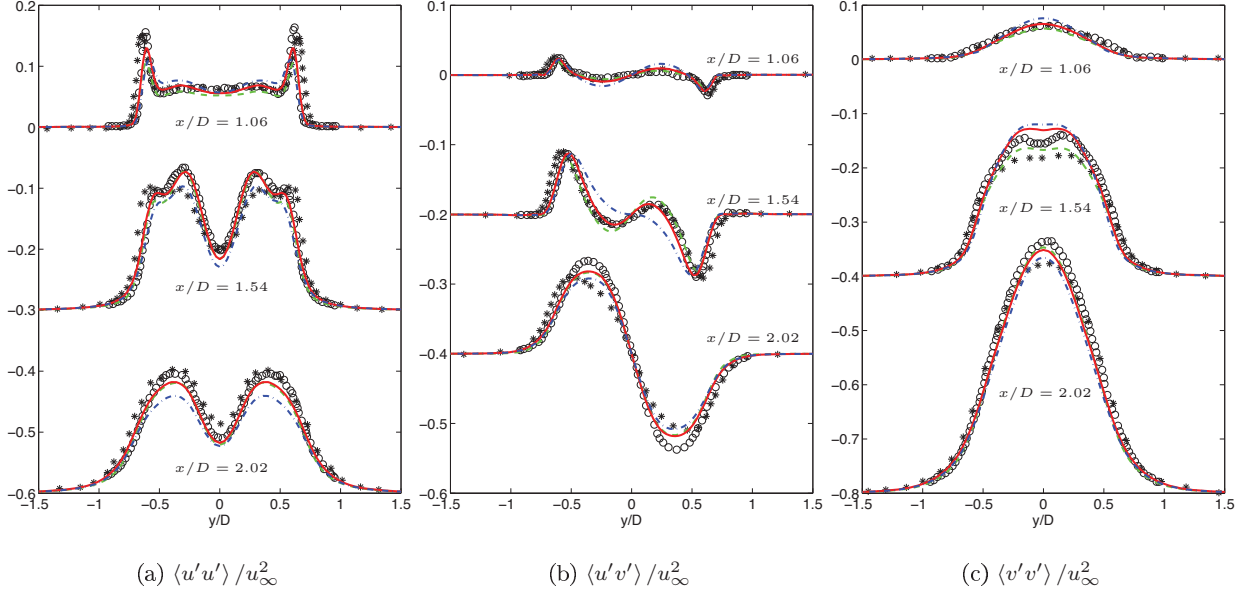


Figure 12: Normalized Reynolds stress profiles at three locations in the near wake of a circular cylinder at $Re_D = 3900$. In (a), profiles at $x/D = 1.54$ and $x/D = 2.02$ are vertically shifted down by 0.3 and 0.6, respectively. In (b), profiles at $x/D = 1.54$ and $x/D = 2.02$ are vertically shifted down by 0.2 and 0.4, respectively. In (c), profiles at $x/D = 1.54$ and $x/D = 2.02$ are vertically shifted down by 0.4 and 0.8, respectively. See Fig. 11 for labels.

than that of Parnaudeau et al. [20] from the cylinder to $x/D \simeq 1.2$ in the wake.

Compared to v^2 - f DES and dynamic-Smagorinsky LES, SA-DDES predicts rather smooth profiles of V , $\langle u'u' \rangle$, and $\langle u'v' \rangle$ in the near wake. This is related to large mean eddy viscosity $\langle \nu_t \rangle$ produced by SA-DDES, as shown in Fig. 13. v^2 - f DES and dynamic-Smagorinsky LES generate $\langle \nu_t \rangle \sim \nu$ in the near wake around $1 \lesssim x/D \lesssim 2$, whereas $\langle \nu_t \rangle \sim 2\nu$ from SA-DDES. Interestingly, both v^2 - f DES and SA-DDES have $\langle \nu_t \rangle$ peaks around $x/D \simeq 1.5$ close to the centerline in the near wake, whereas the dynamic Smagorinsky has two symmetric peaks at $x/D \simeq 2.5$ and $y/D \simeq \pm 0.5$. Although v^2 - f DES decreases $\langle \nu_t \rangle$ downstream $x/D \gtrsim 2$, small $\langle \nu_t \rangle$ of v^2 - f DES does not significantly affect mean flow field in the far wake (see Fig. 15).

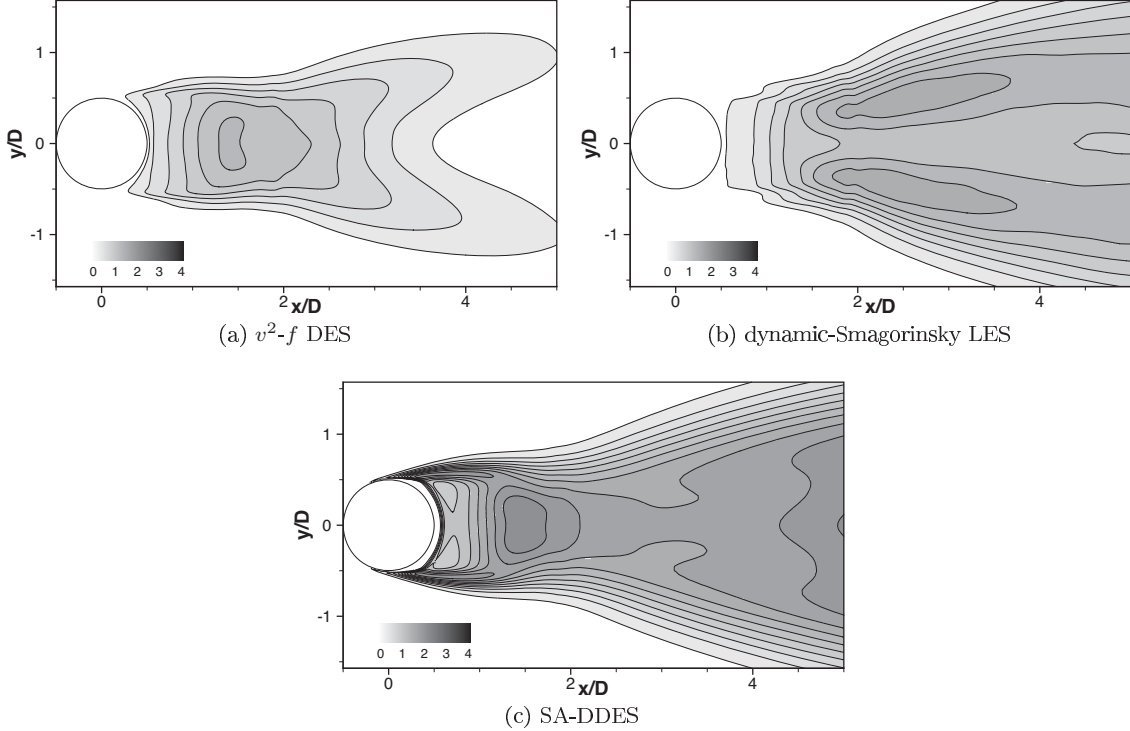


Figure 13: Contours of the normalized mean eddy viscosity $\langle \nu_t \rangle / \nu$. In all plots, $\langle \nu_t \rangle / \nu|_{\min} = 0.2$ and $|\Delta \langle \nu_t \rangle / \nu| = 0.2$.

Fig. 14 shows mean model quantities of the v^2 - f DES model in the flow over the cylinder. Overall, a similar structure is observed in $\langle k \rangle$, $\langle \overline{v^2} \rangle$, and $\langle \epsilon \rangle$; they are symmetric with respect to the centerline with peaks around $x/D \sim 1.5$ near the centerline. $\langle \epsilon_{\text{DES}} \rangle$ is larger than $\langle \epsilon \rangle$, which indicates that the LES mode is active in the wake and that the LES dissipation is larger than the RANS dissipation, as expected. The ratio $\langle \overline{v^2} \rangle / \langle k \rangle$ increases from almost zero near the wall to 0.42 around $x/D \sim 1$ and slowly decreases as the eddy viscosity decreases. Since $\overline{v^2} / k$ does not reach $2/3$ in this simulation, $c_2 \simeq c_{2,0}$ in the whole domain. This phenomena is rather unexpected, although $\langle \overline{v^2} \rangle / \langle k \rangle = 2/3$ is the asymptotic value in the limit of isotropic LES. In order to make $\langle \overline{v^2} \rangle / \langle k \rangle \rightarrow 2/3$ in the wake, the parameters $c_{2,1}$ and $c_{2,2}$ in Eq. (9) probably need to be modified, so c_2 changes more smoothly between $c_{2,0}$ to $2/3$. As mentioned in section 2, these parameters are not optimized. In Fig. 14 (e), $\max(\langle k \rangle, 1e-6)$ is used for $\langle k \rangle$ to avoid $0/0$. The mean function $\langle f \rangle$ has peaks similar to those of $\langle \overline{v^2} \rangle$, which visually shows that f suppresses $\overline{v^2}$ near the wall and produces $\overline{v^2}$ away from the wall.

Mean flow quantities in the far wake are shown in Fig. 15. Only the data of Kravchenko et al.[21] is included as the reference, because Parnaudeau et al.[20] did not include far wake data. Data of Kravchenko et al.[21] has good agreement with the experiment of Ong and Wallace [28]. Similar to the near wake, all

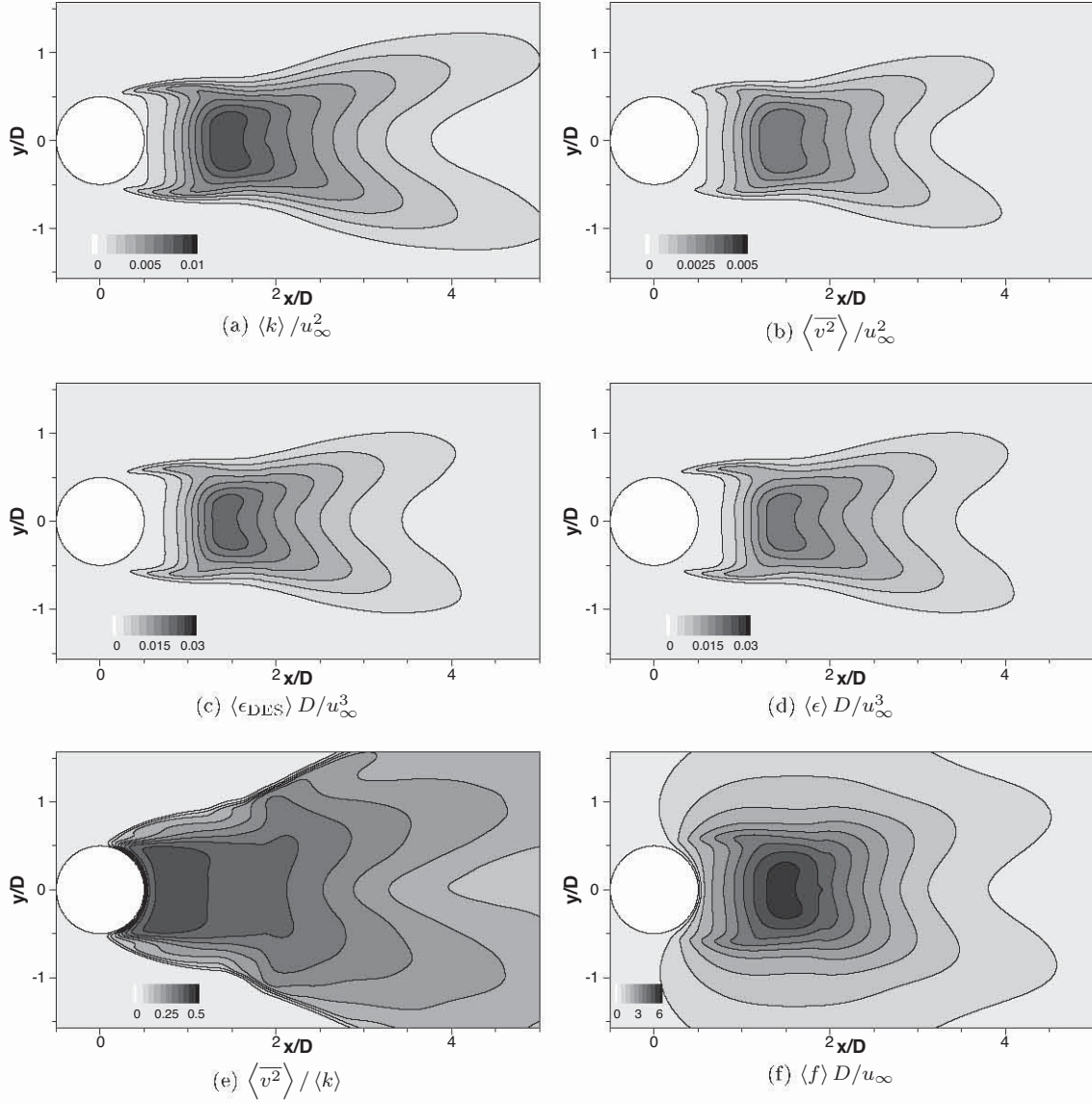


Figure 14: Normalized mean model quantities of the v^2 - f DES model. (a) $|\Delta \langle k \rangle / u_\infty^2| = 0.001$. (b) $|\Delta \langle \overline{v^2} \rangle / u_\infty^2| = 0.005$. (c) $|\Delta \langle \epsilon_{\text{DES}} \rangle D / u_\infty^3| = 0.003$. (d) $|\Delta \langle \epsilon \rangle D / u_\infty^3| = 0.003$. (e) $|\Delta \langle \overline{v^2} \rangle / \langle k \rangle| = 0.05$. (f) $|\Delta \langle f \rangle D / u_\infty| = 0.6$.

three turbulence models give the acceptable profiles. v^2 - f DES has slightly better agreement in $\langle u'v' \rangle$ than the others, whereas dynamic Smagorinsky is slightly better for $\langle u'u' \rangle$. Based on both $\langle u'u' \rangle$ and $\langle v'v' \rangle$, this far-wake flow is still far from isotropic turbulence.

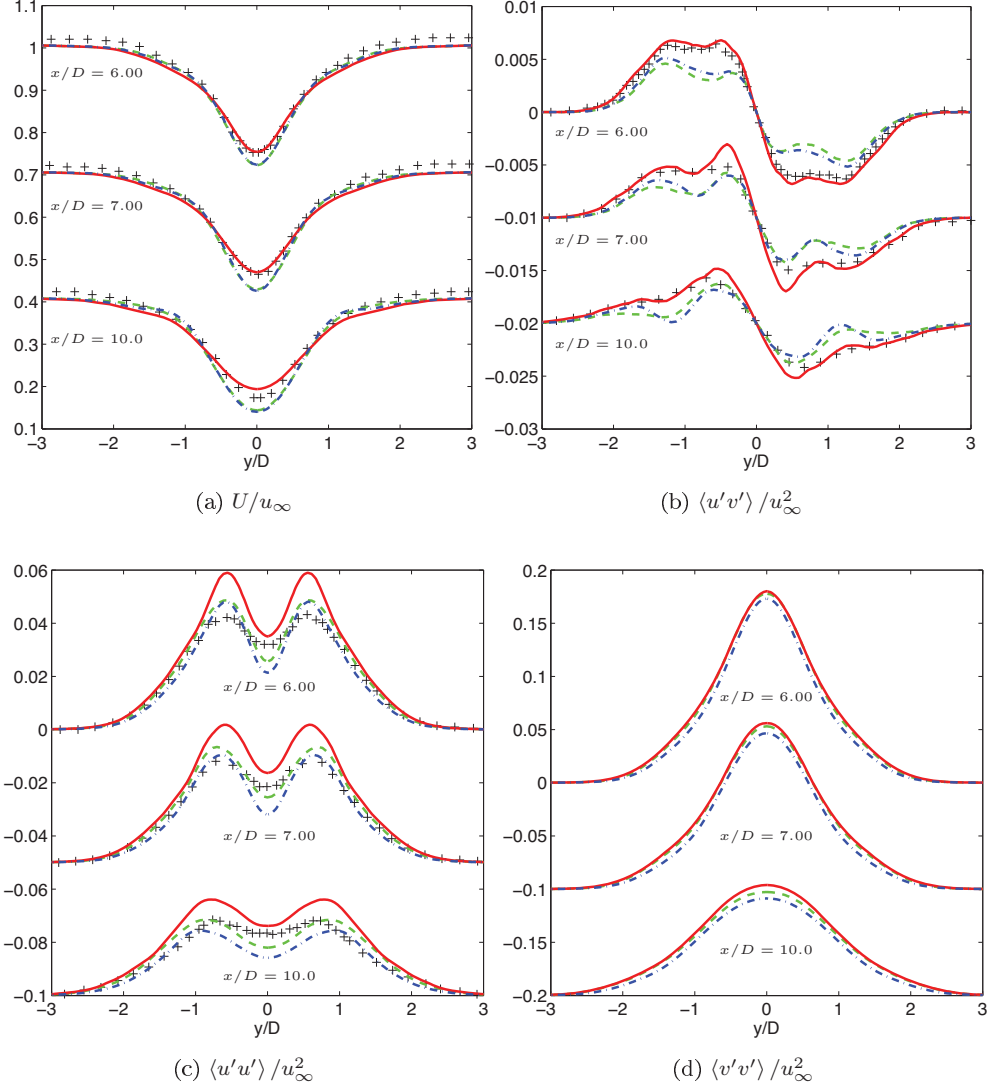


Figure 15: Mean streamwise velocity (a), Reynolds shear stress $\langle u'v' \rangle$ (b), and normal stresses $\langle u'u' \rangle$ (c) and $\langle v'v' \rangle$ (d), at three locations in the far wake of a circular cylinder at $Re_D = 3900$. In (a), profiles at $x/D = 7.00$ and $x/D = 10.0$ are vertically shifted down by 0.3 and 0.6, respectively. In (b), profiles at $x/D = 7.00$ and $x/D = 10.0$ are vertically shifted down by 0.01 and 0.02, respectively. In (c), profiles at $x/D = 7.00$ and $x/D = 10.0$ are vertically shifted down by 0.05 and 0.1, respectively. In (d), profiles at $x/D = 7.00$ and $x/D = 10.0$ are vertically shifted down by 0.1 and 0.2, respectively. See Fig. 11 for labels.

4 Conclusions and Future Work

DES based on the v^2 - f model was proposed, implemented in the unstructured incompressible code CDP, and tested for isotropic turbulence and flow around a circular cylinder. Three coefficients in the elliptic relaxation equation of the original RANS model are modified in order that $\overline{v^2}$ is statistically $(2/3)k$ in the limit of isotropic turbulence. This allows the v^2 - f DES formulation to reduce to sgs- k LES in this limit. The DES coefficient C_{DES} determined from isotropic turbulence is $C_{\text{DES}} = 0.8$.

Flow around a circular cylinder at $Re_D = 3900$ is simulated with the proposed v^2 - f DES along with SA-based DDES and the dynamic Smagorinsky model. Since at this Reynolds number turbulence occurs after separation, this case tests only the LES mode. v^2 - f DES reproduces not only instantaneous large structures in the wake but also mean flow fields as observed in previous experimental and numerical studies. Dynamic-Smagorinsky LES and SA-DDES are performed for direct comparison among the three models. Overall, all three turbulence models accurately predict integral quantities, mean velocity profiles and turbulence intensity profiles in the wake. Although some quantities are predicted slightly better by a particular model, the difference is not sufficient enough for one to assert that one of the models is the best. For subcritical Reynolds numbers, it is expected that all DES models should behave like a good LES model. Further numerical tests, specifically, for wall bounded internal flows where turbulent separation and reattachment locations are more difficult to predict, remain for future study.

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